COSET MODELS AND DIFFERENTIAL GEOMETRY [*]

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ABSTRACT

String propagation on a curved background defines an embedding problem of surfaces in differential geometry. Using this, we show that in a wide class of backgrounds the classical dynamics of the physical degrees of freedom of the string involves 2–dim σ–models corresponding to coset conformal field theories.

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Coset models have been used in string theory for the construction of classical vacua, either as internal theories in string compactification or as exact conformal field theories representing curved spacetimes. Our primary aim in this note, based on [1], is to reveal their usefulness in a different context by demonstrating that certain classical aspects of constraint systems are governed by 2–dim σ–models corresponding to some specific coset conformal field theories. In particular, we will examine string propagation on arbitrary curved backgrounds with Lorentzian signature which defines an embedding problem in differential geometry, as it was first shown for 4–dim Minkowski space by Lund and Regge [2]. Choosing, whenever possible, the temporal gauge one may solve the Virasoro constraints and hence be left with $D-2$ coupled non–linear differential equations governing the dynamics of the physical degrees of freedom of the string. By exploring their integrability properties, and considering as our Lorentzian background D–dim Minkowski space or the product form $R \otimes K_{D-1}$, where K_{D-1} is any WZW model for a semi–simple compact group, we will establish connection with the coset model conformal field theories $SO(D-1)/SO(D-2)$. This universal behavior irrespectively of the particular WZW model K_{D-1} is rather remarkable, and sheds new light into the differential geometry of embedding surfaces using concepts and field variables, which so far have been natural only in conformal field theory.

Let us consider classical propagation of closed strings on a D–dim background that is the direct product of the real line R (contributing a minus in the signature matrix) and a general manifold (with Euclidean signature) K_{D-1}. We will denote $\sigma^{\pm} = \frac{1}{2}(\tau \pm \sigma)$, where τ and σ are the natural time and spatial variables on the world–sheet Σ. Then, the 2–dim σ–model action is given by

$$S = \frac{1}{2} \int_{\Sigma} (G_{\mu\nu} + B_{\mu\nu}) \partial_+ y^{\mu} \partial_- y^{\nu} - \partial_+ y^0 \partial_- y^0 \ , \qquad \mu, \nu = 1, \dots, D-1 \ , \qquad (1)$$

where G, B are the non–trivial metric and antisymmetric tensor fields and are independent of y^0. The conformal gauge, we have implicitly chosen in writing down (1), allows us to further set $y^0 = \tau$ (temporal gauge). Then we are left with the $D-1$ equations of motion corresponding to the y^{μ}'s, as well as with the Virasoro constraints

$$G_{\mu\nu} \partial_{\pm} y^{\mu} \partial_{\pm} y^{\nu} = 1 \ , \qquad (2)$$

which can be used to further reduce the degrees of freedom by one, thus leaving only the $D-2$ physical ones. We also define an angular variable θ via the relation

$$G_{\mu\nu} \partial_+ y^{\mu} \partial_- y^{\nu} = \cos\theta \ . \qquad (3)$$

In the temporal gauge we may restrict our analysis entirely on K_{D-1} and on the projection of the string world–sheet Σ on the $y^0 = \tau$ hyperplane. The resulting 2–dim surface S has Euclidean signature with metric given by the metric $G_{\mu\nu}$ on K_{D-1} restricted on S. Using (2), (3) we find that the corresponding line element reads

$$ds^2 = d\sigma^{+2} + d\sigma^{-2} + 2\cos\theta d\sigma^+ d\sigma^- \ . \qquad (4)$$

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In general, determining the classical evolution of the string is equivalent to the problem of determining the 2–dim surface that it forms as it moves. Phrased in purely geometrical terms this is equivalent, in our case, to the embedding problem of the 2–dim surface S with metric (4) into the $(D-1)$–dim space K_{D-1}. The solution requires that a complete set of $D-1$ vectors tangent and normal to the surface S as functions of σ_+ and σ_- is found. In our case the 2 natural tangent vectors are $\{\partial_+ y^\mu, \partial_- y^\mu\}$, whereas the remaining $D-3$ normal ones will be denoted by $\{\xi_\sigma^\mu, \sigma = 3, 4, \ldots, D-1\}$. These vectors obey first order partial differential equations [3] that depend, as expected, on the detailed structure of K_{D-1}. Since we are only interested in some universal aspects we will solely restrict to the corresponding <u>compatibility</u> equations. In general, these involve the Riemann curvatures for the metrics of the two spaces S and K_{D-1}, as well as the second fundamental form with components $\Omega_{\pm\pm}^\sigma$, $\Omega_{+-}^\sigma = \Omega_{-+}^\sigma$ and the third fundamental form ($\equiv$ torsion) with components $\mu_\pm^{\sigma\tau} = -\mu_\pm^{\tau\sigma}$ [3]. It turns out that the $D-1$ classical equations of motion for (1) (in the gauge $y^0 = \tau$) and the two constraints (2) completely determine the components of the second fundamental form Ω_{+-}^σ [1]. In what follows we will also use instead of $\mu_\pm^{\sigma\tau}$ a modified, by a term that involves $H_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]}$, torsion $M_\pm^{\sigma\tau}$ [1]. Then the compatibility equations for the remaining components $\Omega_{\pm\pm}^\sigma$ and $M_\pm^{\sigma\tau}$ are [1]:

$$\Omega_{++}^\tau \Omega_{--}^\tau + \sin\theta \partial_+ \partial_- \theta = -R_{\mu\nu\alpha\beta}^+ \partial_+ y^\mu \partial_+ y^\alpha \partial_- y^\nu \partial_- y^\beta \ , \tag{5}$$

$$\partial_\mp \Omega_{\pm\pm}^\sigma - M_\mp^{\tau\sigma} \Omega_{\pm\pm}^\tau - \frac{1}{\sin\theta} \partial_\pm \theta \Omega_{\mp\mp}^\sigma = R_{\mu\nu\alpha\beta}^\mp \partial_\pm y^\mu \partial_\pm y^\alpha \partial_\mp y^\beta \xi_\sigma^\nu \ , \tag{6}$$

$$\partial_+ M_-^{\sigma\tau} - \partial_- M_+^{\sigma\tau} - M_-^{\rho[\sigma} M_+^{\tau]\rho} + \frac{\cos\theta}{\sin^2\theta} \Omega_{++}^{[\sigma} \Omega_{--}^{\tau]} = R_{\mu[\beta\alpha]\nu}^- \partial_+ y^\mu \partial_- y^\nu \xi_\sigma^\alpha \xi_\tau^\beta \ , \tag{7}$$

where the curvature tensors and the covariant derivatives $D_\mu^\pm$ are defined using the generalized connections that include the string torsion $H_{\mu\nu\rho}$.[1] Equations (5)–(7) are generalizations of the Gauss–Codazzi and Ricci equations for a surface immersed in Euclidean space. For $D \geq 5$ there are $\frac{1}{2}(D-3)(D-4)$ more unknown functions (θ, $\Omega_{\pm\pm}^\sigma$ and $M_\pm^{\sigma\tau}$) than equations in (5)–(7). However, there is an underlying gauge invariance [1] which accounts for the extra (gauge) degrees of freedom and can be used to eliminate them (gauge fix).

Making further progress with the embedding system of equations (5)–(7) as it stands seems a difficult task. This is due to the presence of terms depending explicitly on $\partial_\pm y^\mu$ and ξ_σ^μ, which can only be determined by solving the actual string evolution equations. Moreover, a Lagrangian from which (5)–(7) can be derived as equations of motion is also lacking. Having such a description is advantageous in determining the operator content of the theory and for quantization. Rather remarkably, all of these problems can be simultaneously solved by considering for K_{D-1} either flat space with zero torsion or any WZW model based on a semi–simple compact group G, with $\dim(G) = D-1$. This is due to the identity

$$R_{\mu\nu\alpha\beta}^\pm = 0 \ , \tag{8}$$

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[1]We have written (7) in a slightly different form compared to the same equation in [1] using the identity $D_\mu^- H_{\nu\alpha\beta} = R_{\mu[\nu\alpha\beta]}^-$.

which is valid not only for flat space with zero torsion but also for all WZW models [4]. Then we completely get rid of the bothersome terms on the right hand side of (5)–(7).[2] In is convenient to extend the range of definition of Ω^σ_{++} and $M^{\sigma\tau}_\pm$ by appending new components defined as: $\Omega^2_{++} = \partial_+\theta$, $M^{\sigma 2}_+ = \cot\theta\Omega^\sigma_{++}$ and $M^{\sigma 2}_- = -\Omega^\sigma_{--}/\sin\theta$. Then equations (5)–(7) can be recast into the suggestive form

$$\partial_-\Omega^i_{++} + M^{ij}_-\Omega^j_{++} = 0 \ , \tag{9}$$

$$\partial_+M^{ij}_- - \partial_-M^{ij}_+ + [M_+, M_-]^{ij} = 0 \ , \tag{10}$$

where the new index $i = (2,\sigma)$. Equation (10) is a zero curvature condition for the matrices $M_\pm$ and it is locally solved by $M_\pm = \Lambda^{-1}\partial_\pm\Lambda$, where $\Lambda \in SO(D-2)$. Then (9) can be cast into equations for $Y^i = \Lambda^{i2}\sin\theta$ [1]

$$\partial_-\left(\frac{\partial_+Y^i}{\sqrt{1-\vec{Y}^2}}\right) = 0 \ , \qquad i = 2, 3, \ldots, D-1 \ . \tag{11}$$

These equations were derived before in [5], while describing the dynamics of a free string propagating in D–dimensional *flat* space–time. It is remarkable that they remain unchanged even if the flat $(D-1)$–dim space–like part is replaced by a curved background corresponding to a general WZW model. Nevertheless, it should be emphasized that the actual evolution equations of the normal and tangent vectors to the surface are certainly different from those of the flat space free string and can be found in [1].

As we have already mentioned, it would be advantageous if (11) (or an equivalent system) could be derived as classical equations of motion for a 2–dim action of the form

$$S = \frac{1}{2\pi\alpha'}\int(g_{ij} + b_{ij})\partial_+x^i\partial_-x^j \ , \qquad i,j = 1, 2, \ldots, D-2 \ . \tag{12}$$

The above action has a $(D-2)$–dim target space and only models the non–trivial dynamics of the physical degrees of freedom of the string which itself propagates on the background corresponding to (1) which has a D–dim target space. The construction of such an action involves a non–local change of variables and is based on the observation [1] that (11) imply chiral conservation laws, which are the same as the equations obeyed by the classical parafermions for the coset model $SO(D-1)/SO(D-2)$ [6].

We recall that the classical σ–model action corresponding to a coset G/H is derived from the associated gauged WZW model and the result is given by

$$S = I_0(g) + \frac{1}{\pi\alpha'}\int \text{Tr}(t^ag^{-1}\partial_+g)M^{-1}_{ab}\text{Tr}(t^a\partial_-gg^{-1}) \ , \qquad M^{ab} \equiv \text{Tr}(t^agt^bg^{-1} - t^at^b) \ , \tag{13}$$

where $I_0(g)$ is the WZW action for a group element $g \in G$ and $\{t^A\}$ are representation matrices of the Lie algebra for G with indices split as $A = (a, \alpha)$, where $a \in H$ and $\alpha \in G/H$. We have also assumed that a unitary gauge has been chosen by fixing

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[2]Actually, the same result is obtained by demanding the weaker condition $R^-_{\mu\nu\alpha\beta} = R^-_{\mu\alpha\nu\beta}$, but we are not aware of any examples where these weaker conditions hold.

$\dim(H)$ variables among the total number of $\dim(G)$ parameters of the group element g. Hence, there are $\dim(G/H)$ remaining variables, which will be denoted by x^i. The natural objects generating infinite dimensional symmetries in the background (13) are the classical parafermions (we restrict to one chiral sector only) defined in general as [7]

$$\Psi_+^\alpha = \frac{i}{\pi\alpha'}\mathrm{Tr}(t^\alpha f^{-1}\partial_+ f) \ , \qquad f \equiv h_+^{-1} g h_+ \in G \ , \tag{14}$$

and obeying on shell $\partial_-\Psi_+^\alpha = 0$. The group element $h_+ \in H$ is given as a path order exponential using the on shell value of the gauge field A_+

$$h_+^{-1} = \mathrm{P}e^{-\int^{\sigma^+} A_+} \ , \qquad A_+^a = M_{ba}^{-1}\mathrm{Tr}(t^b g^{-1}\partial_+ g) \ . \tag{15}$$

Next we specialize to the $SO(D-1)/SO(D-2)$ gauged WZW models. In this case the index $a = (ij)$ and the index $\alpha = (0i)$ with $i = 1, 2, \ldots, D-2$. Then the parafermions (14) assume the form (we drop $+$ as a subscript) [6, 1]

$$\Psi^i = \frac{i}{\pi\alpha'}\frac{\partial_+ Y^i}{\sqrt{1-\vec{Y}^2}} = \frac{i}{\pi\alpha'}\frac{1}{\sqrt{1-\vec{X}^2}}(D_+X)^j h_+^{ji} \ ,$$

$$(D_+X)^j = \partial_+ X^j - A_+^{jk} X^k \ , \qquad Y^i = X^j(h_+)^{ji} \ . \tag{16}$$

Thus, equation $\partial_-\Psi^i = 0$ is precisely (11), whereas (13) provides the action (12) to our embedding problem. The relation between the X^i's and the Y^i's in (16) provides the necessary non–local change of variables that transforms (11) into a Lagrangian system of equations. It is highly non–intuitive in differential geometry, and only the correspondence with parafermions makes it natural.

It remains to conveniently parametrize the group element $g \in SO(D-1)$. In the right coset decomposition with respect to the subgroup $SO(D-2)$ we may write [6]

$$g = \begin{pmatrix} 1 & 0 \\ & \\ 0 & h \end{pmatrix} \cdot \begin{pmatrix} b & X^j \\ & \\ -X^i & \delta_{ij} - \frac{1}{b+1}X^i X^j \end{pmatrix} \ , \tag{17}$$

where $h \in SO(D-2)$ and $b \equiv \sqrt{1-\vec{X}^2}$. The range of the parameters in the vector $\vec{X}$ is restricted by $\vec{X}^2 \leq 1$. A proper gauge fixing is to choose the group element h in the Cartan torus of $SO(D-2)$ and then use the remaining gauge symmetry to gauge fix some of the components of the vector $\vec{X}$. If $D = 2N + 3 = \text{odd}$ then we may cast the orthogonal matrix $h \in SO(2N+1)$ and the row vector $\vec{X}$ into the form [1]

$$h = \mathrm{diagonal}\,(h_1, h_2, \ldots, h_N, 1) \ , \qquad h_i = \begin{pmatrix} \cos 2\phi_i & \sin 2\phi_i \\ -\sin 2\phi_i & \cos 2\phi_i \end{pmatrix}$$

$$\vec{X} = (0, X_2, 0, X_4, \ldots, 0, X_{2N}, X_{2N+1}) \ . \tag{18}$$

On the other hand if $D = 2N + 2 = \text{even}$ then $h \in SO(2N)$ can be gauge fixed in a form similar to the one in (18) with the 1 removed. Similarly in the vector $\vec{X}$ there is no

X_{2N+1} component. In both cases the total number of independent variables is $D - 2$, as it should be.

Examples: As a first example we consider the Abelian coset $SO(3)/SO(2)$ [7]. In terms of our original problem it arises after solving the Virasoro constraints for strings propagating on 4–dim Minkowski space or on the direct product of the real line R and the WZW model for $SU(2)$. Using $X_2 = \sin 2\theta$ one finds that

$$A_+ = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (1 - \cot^2 \theta)\partial_+ \phi , \tag{19}$$

and that the background corresponding to (12) has metric [7]

$$ds^2 = d\theta^2 + \cot^2 \theta d\phi^2 . \tag{20}$$

Using (16), the corresponding Abelian parafermions $\Psi_\pm = \Psi_2 \pm i\Psi_1$ assume the familiar form

$$\Psi_\pm = (\partial_+\theta \pm i \cot\theta\partial_+\phi)e^{\mp i\phi \pm i \int \cot^2\theta\partial_+\phi} , \tag{21}$$

up to an overall normalization. An alternative way of seeing the emergence of the coset $SO(3)/SO(2)$ is from the original system of embedding equations (5)–(7) for $D = 4$ and zero curvatures. They just reduce to the classical equations of motion for the 2–dim σ–model corresponding to the metric (20) [2], as it was observed in [8].

Our second example is the simplest non–Abelian coset $SO(4)/SO(3)$ [6]. In our context it arises in string propagation on 5–dim Minkowski space or on the direct product of the real line R and the WZW model based on $SU(2) \otimes U(1)$. Parametrizing $X_2 = \sin 2\theta \cos \omega$ and $X_3 = \sin 2\theta \sin \omega$ one finds that the 3×3 antisymmetric matrix for the $SO(3)$ gauge field A_+ has independent components given by

$$A_+^{12} = -\left(\frac{\cos 2\theta}{\sin^2\theta\cos^2\omega} + \tan^2\omega\frac{\cos^2\theta - \cos^2\phi\cos 2\theta}{\cos^2\theta\sin^2\phi}\right)\partial_+\phi - \cot\phi\tan\omega\tan^2\theta\partial_+\omega ,$$

$$A_+^{13} = \tan\omega\frac{\cos^2\theta - \cos^2\phi\cos 2\theta}{\cos^2\theta\sin^2\phi}\partial_+\phi + \cot\phi\tan^2\theta\partial_+\omega , \tag{22}$$

$$A_+^{23} = \cot\phi\tan\omega\frac{\cos 2\theta}{\cos^2\theta}\partial_+\phi - \tan^2\theta\partial_+\omega .$$

Then, the background metric for the action (12) governing the dynamics of the 3 physical string degrees of freedom is [6]

$$ds^2 = d\theta^2 + \tan^2\theta(d\omega + \tan\omega\cot\phi d\phi)^2 + \frac{\cot^2\theta}{\cos^2\omega}d\phi^2 , \tag{23}$$

and the antisymmetric tensor is zero. The parafermions of the $SO(4)/SO(3)$ coset are non–Abelian and are given by (16) with some explicit expressions for the covariant derivatives [1]. In addition to the two examples above, there also exist explicit results for the coset $SO(5)/SO(4)$ [9]. This would correspond in our context to string propagation on a 6–dim Minkowski space or on the background R times the $SU(2) \otimes U(1)^2$ WZW model.

An obvious extension one could make is to consider the same embedding problem but with Lorenzian instead of Euclidean backgrounds representing the "spatial" part K_{D-1}. This would necessarily involve σ–models for cosets based on non–compact groups. The case for $D = 4$ has been considered in [10]. It is interesting to consider supersymmetric extensions of the present work in connection also with [11]. In addition, formulating classical propagation of p–branes on curved backgrounds as a geometrical problem of embedding surfaces (for work in this direction see [12]) and finding the $p + 1$–dim σ–model action (analog of (12) for strings ($p = 1$)) that governs the dynamics of the physical degrees of freedom of the p–brane is an open interesting problem.

The techniques we have presented in this note can also be used to find the Lagrangian description of the symmetric space sine–Gordon models [13] which have been described as perturbations of coset conformal field theories [14]. Hence, the corresponding parafermion variables will play the key role in such a construction.

Finally, an interesting issue is the quantization of constrained systems. Quantization in string theory usually proceeds by quantizing the unconstrained degrees of freedom and then imposing the Virasoro constraints as quantum conditions on the physical states. However, in the present framework the physical degrees of freedom should be quantized directly using the quantization of the associated parafermions. Quantization of the $SO(3)/SO(2)$ parafermions has been done in the seminal work of [15], whereas for higher dimensional cosets there is already some work in the literature [16]. A related problem is also finding a consistent quantum theory for vortices. This appears to have been the initial motivation of Lund and Regge (see [2]).

Acknowledgments

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